Solving The Dynamic Volatility Fitting Problem:

A Deep Reinforcement Learning Approach in Continuous State Action Spaces

Flow Equity Derivatives.

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Volatility Fitting Problem and why ?

Mark-to-market quoted price P: $\left\{P^{Ask}_{Call/Put}, P^{Bid}_{Call/Put}\right\} \xrightarrow{\phi^{\text{Model}}} \left\{\sigma^{Ask}_{Call/Put}, \sigma^{Bid}_{Call/Put}\right\} \xrightarrow{\phi} \left\{\sigma^{Ask}, \sigma^{Bid}\right\}$ for different moneyness $(\kappa_1, ..., \kappa_n)$ where ϕ is a non unique mapping and $\phi^{\text{Model}}=$ Black-Scholes for Ex.

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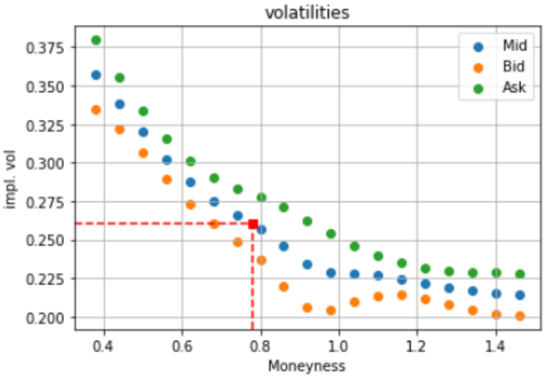

The volatility fitting problems aims to match the market implied volatility. It is used:

- To price structured products with volatility exposure.
- For risk management
- etc.

Encoding the market data:

Many methods in the industry to encode the implied volatility information [3]

- Non-Parametric grid of impl. Vol then interpolation via simple spline.
- Modelling the implied density of the asset directly (Carr and Wu approach)
- Use a diffusion style model ((L)SV models)

[3]flexible and compatible with arbitrage constraints (call spread, butterfly and calendar spread)

[4]Ex: Stochastic volatility inspired (SVI) parameterization by Gatheral and Antoine Jacquier: $\vec{\theta} = (a, b, \rho, m, \sigma)$

$$\Psi_{\vec{\theta}}^{SVI}(k)^2 = a + b(\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2})$$

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- Surface-Level Parametrizations [4]

$\rightarrow$ In this setting, we model the vol with a parametric functional $\Psi_{\vec{\theta}}^{vol}(\kappa)$ for a given maturity.

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Traditional approach to volatility fitting with Parametrisation:

$\Rightarrow$ Choice of the target objective function:[5] : Among Others, the squared-error loss

$$\xi(\vec{\theta}_{t_i}) := \sqrt{\frac{\sum_{j=1}^{n} w_j \ell(t_i, \kappa_j)^2}{\sum_{j=1}^{n} w_j}} \text{ with } \ell(t_i, \kappa_j) := \frac{\sigma^{Mid}(t_i, \kappa_j) - \Psi^{vol}_{\vec{\theta}_{t_i}}(\kappa_j)}{\sigma^{\text{spread}}(t_i, \kappa_j)}, \quad w_j = \text{Black-Scholes Vega} \tag{1}$$

$\Rightarrow$ Given vol. surf. param. $\Psi_{\vec{\theta}_{t_i}}$ and $\xi(\vec{\theta}_{t_i})$, adjust $\vec{\theta}_{t_i}$ to match the mid market observable impl. vol. σ^{Mid}

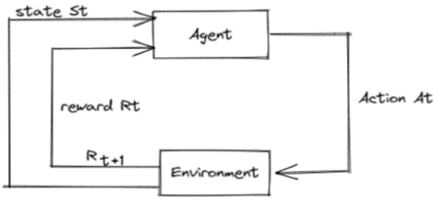
Drawbacks:

- Large set of conditions (liquidity, macro-events, term-structure, stability)$\rightarrow$ optimizer too heavy to run
- Imply an optimisation at each move of the market (speed and reliability).
- Cannot re-use past accumulated experiences (every output is flashed out).
- Do not handle very well a case with market data limitations and irregularities.

$\Rightarrow$ We rely on a fully AI-based method to learn market regimes and discover optimal behaviours.

[5] Practitioners aim at positioning the model implied volatility at the mid.

Towards a Novel Framework: Background on Reinforcement Learning.



An agent reinforced with awards based on its interaction with the environment would learn to operate optimally in that environment to maximize the rewards.

- The agent observes a current state $s_t \in \mathcal{S} \subset \mathbb{R}^d$, takes an action $a_t \in \mathcal{A} \subset \mathbb{R}^n$ drawn from the policy $\Pi \supset \pi : \mathcal{S} \to \mathcal{A}$ (resp. $\mathcal{P}(\mathcal{A})$) and receives a reward $r_t = r(a_t, s_t)$.

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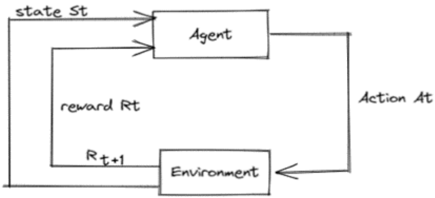

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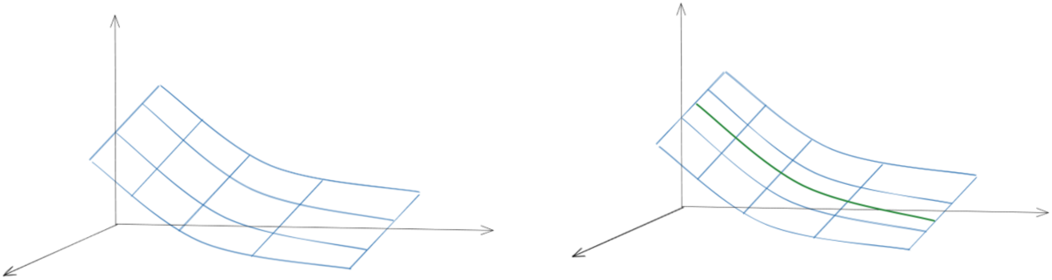
- Set $\gamma \in [0,1]$, define the Return from a state from following the policy π, $R_t = \sum_{i=t}^{T} \gamma^{(i-t)} r(s_i, a_i)$
- state-action value function $Q^\pi(s,a) = \mathbb{E}^\pi[R_t | s_t = s, a_t = a]$.
- Mathematical formulation of the problem: $\boxed{\pi^* = \arg\max_{\pi \in \Pi} Q^\pi}$

Problem Statement in Reinforcement Learning:

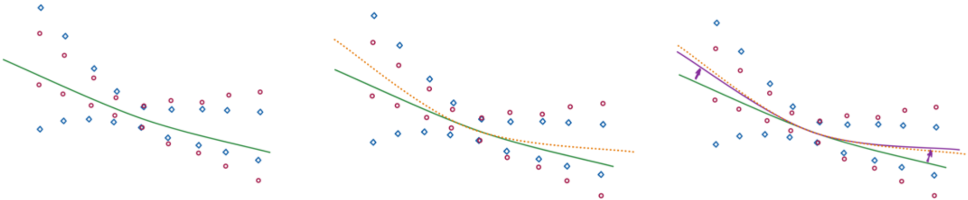


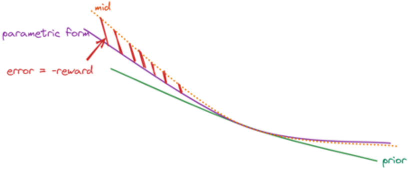
(a) (b)

- Fixed expiry T, and observe the market at time t_i, i.e. a collection $\left\{\sigma^{Ask}_{Call/Put}(t_i,\kappa_j),\sigma^{Bid}_{Call/Put}(t_i,\kappa_j)\right\}_{j\in[1,\ldots,n]}$ of markets quotes for different moneyness $(\kappa_1,\ldots,\kappa_n)$.
- we consider a parametrisation $\Psi^{vol}_{\vec{\theta}}$ of the volatility slice with a vector of K parameters.

RL Framework for Fitting:

- Starting from a prior parametrized volatility surface $\Psi^{vol}_{\vec{\theta}_{t_i}}$ (i.e. the latest fit is the prior input).





Advanced Fitting: a reinforcement learning approach

Goodness of Fit in RL-Based Volatility Fitting

- Mean squared error (MSE) [6] defined by :

$$\xi(\vec{\theta}_{t_i}) := \sum_{j=1}^{n} \left(\sigma^{Mid}(t_i, \kappa_j) - \Psi^{vol}_{\vec{\theta}_{t_i}}(k_i) \right)^2 \tag{3}$$

- The problem is to find the shifted vector $\Delta\vec{\theta}_{t_i} = \left(\Delta\theta^1_{t_i}, \ldots, \Delta\theta^K_{t_i}\right)$ to bump the old parameters $\vec{\theta}_{t_i}$ in order to maximize our selected reward function. $\boxed{\Delta\vec{\theta}^*_{t_i} = \arg\max_{\Delta\vec{\theta}_{t_i} \in \mathbb{R}^K} Q^\pi(s_{t_i}, \Delta\vec{\theta}_{t_i}).}$

with $Q^\pi(s_{t_i}, \Delta\vec{\theta}_{t_i}) = \sum_{t=t_i}^{T} \mathbb{E}_{(s_t, a_t)\sim\rho_\pi}\left[\gamma^{(t-t_i)} r(s_t, \Delta\vec{\theta}_t)\right]$ and $\underline{r(s_t, a_t = \Delta\vec{\theta}_t) = -\xi(\theta_t)}$

[6] Alternative. Setting $\sigma^{\text{spread}}(t_i, \kappa_j) := \sigma^{Ask}_{t_i,\kappa_j} - \sigma^{Bid}_{t_i,\kappa_j}$ the volatility spread, we define:

$$\xi(\vec{\theta}_{t_i}) := \sum_{j=1}^{n} \text{vega}_{BS}(\kappa_j) \left(\frac{\sigma^{Mid}(t_i, \kappa_j) - \Psi^{vol}_{\vec{\theta}_{t_i}}(\kappa_j)}{\sigma^{\text{spread}}(t_i, \kappa_j)} \right)^2 \tag{2}$$

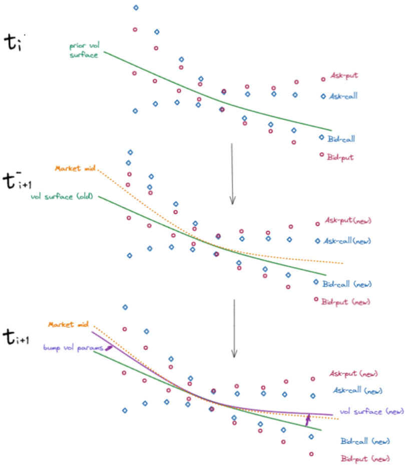
Market environment and Non Linear functional approximation by FCNNs:

1. States are the collection of observable market quotes:
   $s_{t_{i+1}} = \left(\sigma^{Bid}(t_{i+1}, \kappa_j), \sigma^{Ask}(t_{i+1}, \kappa_j), \theta^1_{t_i}, \dots, \theta^K_{t_i}\ \right)_j$
2. Actions are the bumps to apply in order to shift the prior volatility parametrization:
   $a_{t_{i+1}} = \Delta\vec{\theta}_{t_{i+1}} = \vec{\theta}_{t_{i+1}} - \vec{\theta}_{t_i}$
3. Rewards are the opposite of well designed errors.
   $r(s_{t_{i+1}}, a_{t_{i+1}} = \Delta\vec{\theta}_{t_{i+1}}) = -\xi(\theta_{t_{i+1}})$

$$Q^\pi(s_{t_{i+1}}, a_{t_{i+1}}) = -\sum_{k=t_{i+1}}^{T} \mathbb{E}_{(s_k, a_k)\sim\rho_\pi}[\gamma^{(k-t_{i+1})}\xi(\vec{\theta}_k)].$$

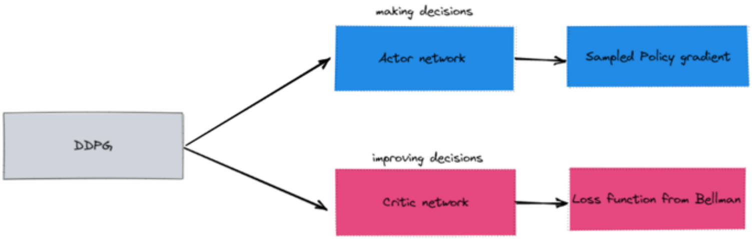
$$\boxed{a^*_{t_{i+1}}(s_{t_{i+1}}) = \arg\max_{a_{t_{i+1}}\in\mathcal{A}} Q^\pi(s_{t_{i+1}}, a_{t_{i+1}})}$$

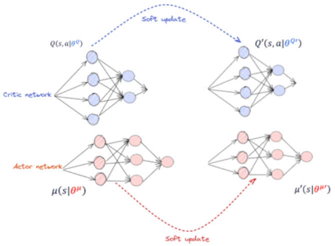
Remark: $\Delta\vec{\theta}$ (the action vectors) can take any values in $\mathbb{R}^K$ space: $\Rightarrow$ continuous actions spaces.



Actor Critic: Deep Deterministic Policy Gradient.(Lillicrap et al., 2015)

- off-policy actor-critic method that learns a deterministic policy in continuous domain.





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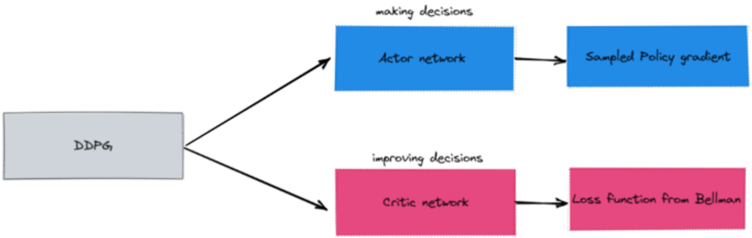

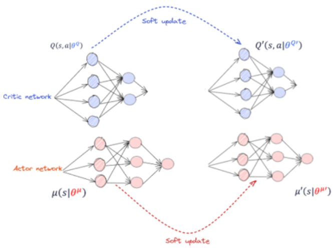

- Action network: DPG= Deep Policy gradient

$$a_t \sim \pi^D(s_t, \theta^\pi) + \epsilon_t \quad \text{with} \quad \epsilon_t \sim \mathcal{N}(0, \sigma_n^2 I_K) \quad \text{and} \quad \sigma_n = \max(\sigma_0(1 - \frac{n}{N})^4, \sigma_{\min})$$

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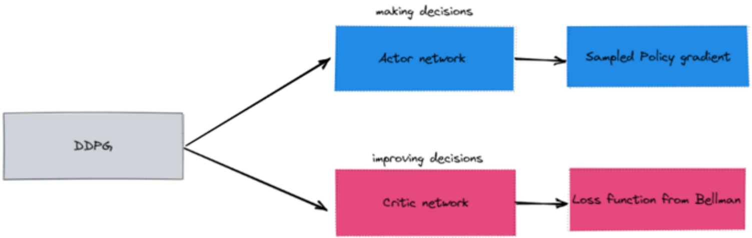

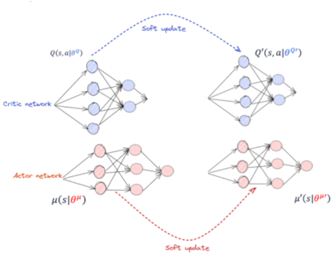

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- Critic network: Q-Learning and Bellman equation.

$$\begin{cases} R_t = \sum_{i=t}^{T} \gamma^{(i-t)} r(s_i, a_i) \\ Q^\pi(s_t, a_t) = \mathbb{E}[R_t | s_t, a_t] \end{cases} \Rightarrow L(\theta^Q) = \mathbb{E}[(Q^\pi(s_t, a_t; \theta^Q) - [r(s_t, a_t) + \gamma Q^\pi(s_{t+1}, a_{t+1}; \theta^Q)])^2]$$

Numerical results: Convergence of DDPG Algorithm.

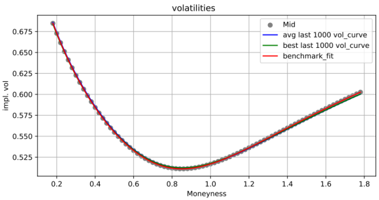
- To assess the performance of the RL algorithms for the volatility fitting, we consider toy problems of increasing complexity and perform a Monte Carlo on 5 differents random seeds:

1. Static scenario:
    - we explore the scenarios where the market conditions remain constant.

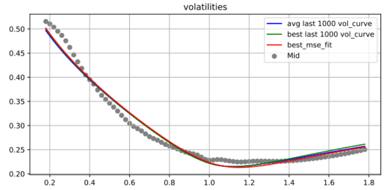
Play Video

Static Market, DDPG algorithm, MSE Reward:

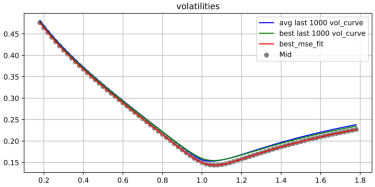
Below, best response of the agent amongst the last 1000 episodes (*blue*) and the mean of the last 1000 volatilities slices (*red*) (a) High Smile (b) Skew (c) High Skew (d) Inverse Smile.

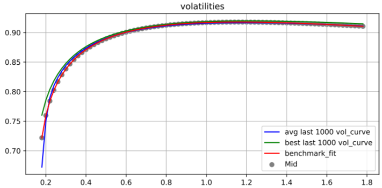


(a)



(b)





DDPG: Convergence and Consistency.

- Performance of the RL algorithms: Monte Carlo on 5 differents random seeds:
  1. Static scenario:
     - RL agents perform as-good as traditional approaches in fitting volatility curve.

Table: DDPG MSE Rewards

type	Skew	High Smile	Inv. Smile
Bench	-0.011913	-0.0000220	-0.0000436
seeds' avg	-0.012145	-0.000163	-0.000315

- Final rewards achieved is close to the one coming from the optimizer.
- Performance is stable across the different market configuration considerered

Quasi Dynamic Scenario: The market evolve freely.

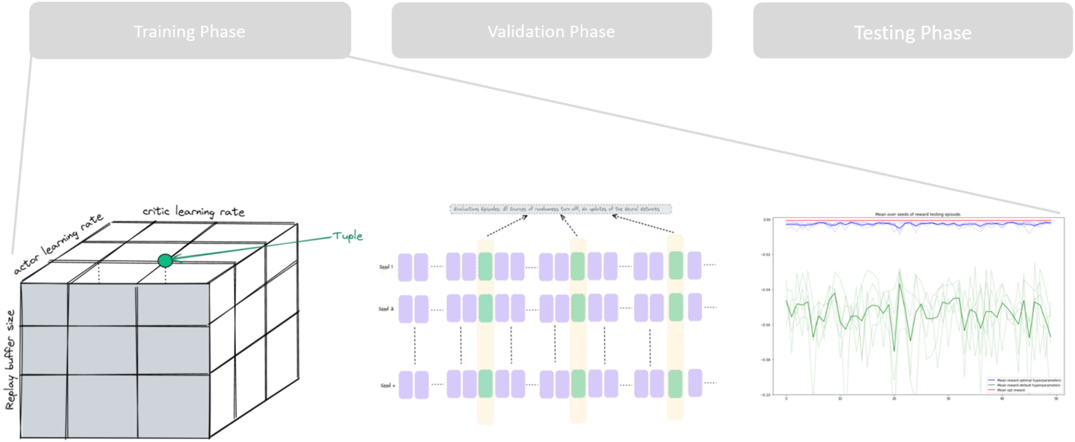


Figure: Training and Consistency over random seeds.

Quasi Dynamic Scenario, MSE Reward

2.Validation Phase: We compare the performance of different successful agents under different seeds.

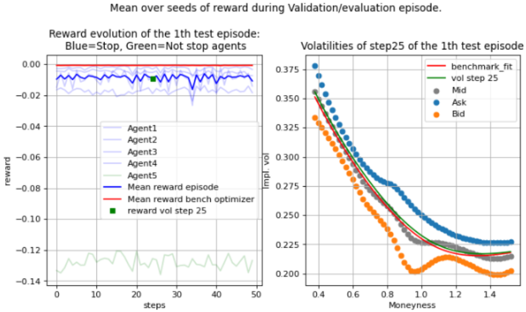


Figure: Evolution of mean episodic rewards(several random seeds) and Implied volatility(of one selected step) for a wide spread stock with **DDPG**.

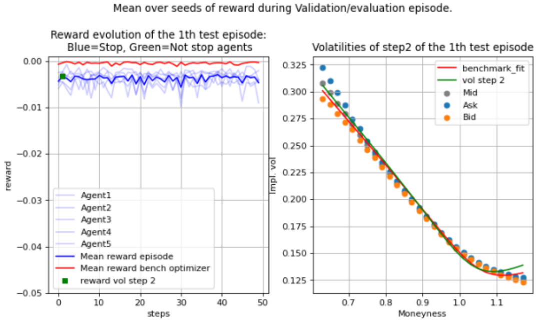


Figure: Evolution of mean episodic rewards(several random seeds) and Implied volatility(of one selected step) for a tight spread stock with **DDPG**.

Quasi Dynamic Scenario, MSE Reward, Testing Phase

3.Testing Phase: we assess the performance of the best agent in a test episode.

Play Video

Vol fitting animation

Conclusion and Prospects

1. Sum up,
   - A wide range of problems can be tackled with Deep RL Actor-Critic methods while neural networks help to approximate the solution function.[7]
   - We showed that DRL algorithms are natively tailored for the volatility fitting problem:
     - native exploration
     - effective catalog of past experiences(re-use past accumulated experiences)
     - handle complex objective function (desk PnL).

[7] optimal execution, portfolio optimization, option pricing and hedging, market making, smart order routing

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2. Future work could be:
   - Reflects stylistic effects in volatility surface term-structure.
   - Careful rewards shaping and environment design.

[7] optimal execution, portfolio optimization, option pricing and hedging, market making, smart order routing

Thanks for your attention!

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